B.Sc. Part II (Hons) 4th Paper

Vectors contd.

Q. Prove that $\operatorname{div} \hat{r} = \frac{2}{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

Soln

We know that

$$\hat{r} = \frac{\vec{r}}{r} \Rightarrow \hat{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$

$$\Rightarrow \hat{r} = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k} \quad \text{------ (1)}$$

Let $f_1 = \frac{x}{r}$, $f_2 = \frac{y}{r}$, $f_3 = \frac{z}{r}$

$$\text{(1)} \Longrightarrow \hat{r} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k} \quad \text{------ (2)}$$

Now $\operatorname{div} \hat{r} = \nabla \cdot \hat{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \hat{r}$

$$\Rightarrow LHS = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\vec{i} + f_2\vec{j} + f_3\vec{k}\right)$$

$$LHS = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \text{------ (3)}$$

Now, $\frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$ [using (1)]

$$= \frac{r \cdot 1 - x \cdot \frac{\partial r}{\partial x}}{r^2}$$

$$\Rightarrow \frac{\partial f_1}{\partial x} = \frac{r - x\frac{\partial r}{\partial x}}{r^2}$$

$\because \frac{\partial r}{\partial z}$

$$\Rightarrow \frac{\partial f_1}{\partial x} = \frac{r - x \cdot \frac{x}{r}}{r^2}$$

$$\boxed{\begin{array}{l} \because\ r^2 = x^2 + y^2 + z^2 \Rightarrow 2r \cdot \frac{\partial r}{\partial x} = 2x \\ \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \text{ Similarly } \frac{\partial r}{\partial y} = \frac{y}{r}, \\ \frac{\partial r}{\partial z} = \frac{z}{r} \end{array}}$$

$$\left.\begin{array}{l} \Rightarrow \frac{\partial f_1}{\partial x} = \frac{r^2 - x^2}{r^3} \\ \text{Similarly } \frac{\partial f_2}{\partial y} = \frac{r^2 - y^2}{r^3} \\ \text{and } \frac{\partial f_3}{\partial z} = \frac{r^2 - z^2}{r^3} \end{array}\right\} \longleftarrow (4)$$

Putting the values of (4) in (3),

we get

$$LHS = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3}$$

$$= \frac{3r^2 - (x^2 + y^2 + z^2)}{r^3} = \frac{3r^2 - r^2}{r^3} = \frac{2r^2}{r^3}$$

$$= \frac{2}{r} = \underline{\underline{RHS}}$$